

On braided FRT-construction

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Abstract

Fully braided analog of Faddeev-Reshetikhin-Takhtajan construction of quasitriangular bialgebra $A(X, R)$ is proposed. For given pairing C factor-algebra $A(X, R; C)$ is a dual quantum braided group. Corresponding inhomogeneous quantum group is obtained as a result of generalized bosonization. Construction of first order bicovariant differential calculus is proposed.

Key words: Braided category, (dual) quantum braided group, bosonization.

AMS Subject Classifications (1991): 16W30, 17B37, 18D10, 81R50.

1 Introduction and preliminaries

Hopf algebras in braided categories (braided groups) have been extensively studied over the last few years and play an important role in q -deformed physics and mathematics [17],[18]. Examples, applications and the basic theory of braided groups have been introduced and developed by Majid. Some similar concepts arise independently in works of Lyubashenko inspired by results on conformal field theory. Crossed modules over braided Hopf algebras were introduced and studied in [3] and provide a useful technique for investigation of braided Hopf algebras. In particular, crossed product of braided Hopf algebras and generalized bosonization for quantum braided groups are defined in [3]. The theory of Hopf bimodules in braided categories is developed in [4], on grounds of [3]. Application of this theory is an analog of Woronowicz construction of (bicovariant) differential calculi [20] developed in [5] for the case of braided Hopf algebras and quantum braided groups. Quantum braided group defined by Majid [14, 15] is a natural generalization of Drinfel'd's concept of (ordinary) quantum group (quasitriangular Hopf algebra) [7]. Basic examples of coquasitriangular bialgebras $A(R)$ are obtained as a result of Faddeev-Reshetikhin-Takhtajan construction [8] applied to an arbitrary R -matrix. Analog of FRT-construction for anyonic quantum groups is described in [19]. Majid proposed another construction of braided bialgebra $B(R)$ which can be obtained as a

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transmutation [14] of $A(R)$. Algebra $A(R, Z)$ defined in [10] generalizes both $A(R)$ and $B(R)$. In this paper we describe a fully braided analog of FRT-construction of quasitriangular bialgebra $A(X, R)$, where X is an object of an Abelian braided monoidal category \mathcal{C} and $R : X \otimes X \rightarrow X \otimes X$ solution of the braid equation. This construction covers all mentioned above and can be considered as a coordinate-free version of [10]. For a given pairing C we define a factor-algebra $A(X, R; C)$ which is a dual quantum braided group. This is an analog of construction of quantum simple Lie groups of type B, C, D in [8]. Majid's definition of braided vectors $V(R)$ is simply reformulated to our more abstract setting. In particular, our algebra $V(X, R)$ is also a quantum braided group in the category of comodules over $A(X, R)$. Quantized analogs of inhomogeneous linear groups are studied in many papers (see [1, 6] and references therein). Generalized bosonization construction [3] allows us to define quantum braided group $A(X, R) \ltimes V(X, R)$. We propose construction of a first order bicovariant differential calculus on dual quantum braided group A related with any comodule X over A . In our special case $A = A(X, R)$ this is a generalization of construction [12].

In the rest of this part we give necessary preliminary results. The main results of the paper are presented in the second part.

1.1 We will suppose that \mathcal{C} is an *Abelian and braided (monoidal) category* with tensor product \otimes , unit object $\underline{1}$ and braiding Ψ (without loss of generality by Mac Lane's coherence theorem we will assume that underlying monoidal category is strict, i.e. the functors $- \otimes (- \otimes -)$ and $(- \otimes -) \otimes -$ coincide and $\underline{1} \otimes X = X = X \otimes \underline{1}$). Compatibility conditions between tensor product and Abelian structure are the following [5]: functors $(-) \otimes X$ and $X \otimes (-)$ are right exact for any object X (this assumption is true if the category is closed); for any epimorphisms $X_i \xrightarrow{f_i} Y_i$, $i = 1, 2$ the diagram

$$\begin{array}{ccc} X_1 \otimes X_2 & \xrightarrow{X_1 \otimes f_2} & X_1 \otimes Y_2 \\ f_1 \otimes X_2 \downarrow & & \downarrow f_1 \otimes Y_2 \\ Y_1 \otimes X_2 & \xrightarrow{Y_1 \otimes f_2} & Y_1 \otimes Y_2 \end{array} \quad (1)$$

is push-out (the right-down part is a colimit of the left-up part). In this case there exists well-behaved constructions of factor-algebra (coalgebra, bialgebra, Hopf algebra) by ideal (coideal, biideal, Hopf ideal). One can define an algebra by generator and relations. We means under 'the ideal generated by relations $f_1 = f_2 : X \rightarrow A$ ' the subobject $\text{Im}(\mu \circ (\mu \otimes A) \circ (A \otimes (f_1 - f_2) \otimes A))$ of algebra A .

1.2 We will work with *graded and filtered algebras in \mathcal{C}* . A (\mathbb{N}) -graded algebra A means a collection of objects A_k , $k \in \mathbb{C}$, multiplications $m_{i,j} : A_i \otimes A_j \rightarrow A_{i+j}$ satisfying associativity conditions and unit $\eta : \underline{1} \rightarrow A_0$. A (\mathbb{N}) -filtered algebra A means a collection of objects $A_{(k)}$, $k \in \mathbb{C}$, such that $A_{(i)}$ is subobject of $A_{(j)}$ if $i < j$, multiplications $m_{(i),(j)} : A_{(i)} \otimes A_{(j)} \rightarrow A_{(i+j)}$ satisfying conditions of associativity

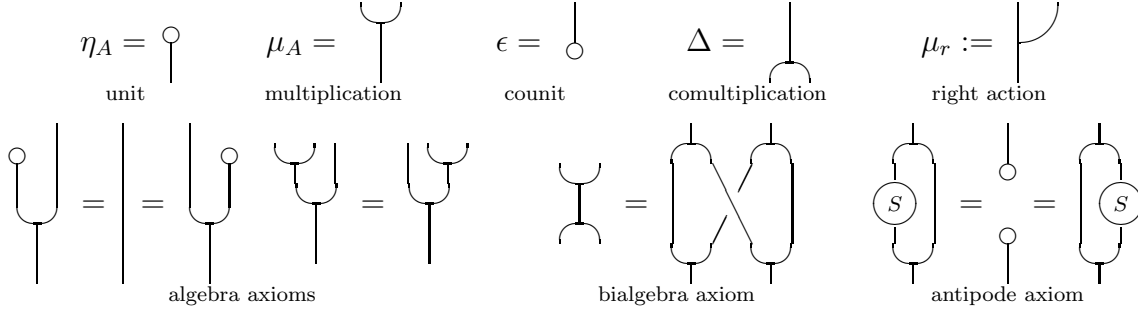


Figure 1: The basic algebraic structures in a braided category

and compatibility with restrictions on subobjects, and unit $\eta : \underline{1} \rightarrow A_{(0)}$. For any graded algebra $\{A_i\}$ the collection $\{A_{(k)} := \bigoplus_{i=0}^k A_i\}$ with natural multiplications is a filtered algebra. As shown in [5] graded or filtered algebra can be considered as a usual algebra in a certain category of 'graded spaces' i.e. functors from a certain category to category \mathcal{C} . This category of 'graded spaces' is again an Abelian braided monoidal category. Similarly graded coalgebras, bialgebras, Hopf algebras can be defined. We will say briefly that graded (filtered) algebra lives in a category \mathcal{C} if its components A_n ($A_{(n)}$) live in \mathcal{C} . See [5] about more details.

1.3 We actively use diagrammatic calculus in braided categories [15, 17] (see [3] about our slight modifications). Morphisms Ψ and Ψ^{-1} are represented by under and over crossing and algebraic information 'flows' along braids and tangles according to functoriality and the coherence theorem for braided categories [11]:

$$\Psi = \begin{array}{c} \diagup \\ \diagdown \end{array} \quad \Psi^{-1} = \begin{array}{c} \diagdown \\ \diagup \end{array} \quad \begin{array}{c} \text{circle } f \\ \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagdown \\ \diagup \\ \text{circle } f \end{array} \quad \begin{array}{c} \text{circle } f \\ \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \\ \text{circle } f \end{array} \quad (2)$$

Fig.1 explains our notations: An *algebra* in a monoidal category \mathcal{C} is an object A equipped with unit $\eta = \eta_A : \underline{1} \rightarrow A$ and multiplication $\mu = \mu_A : A \otimes A \rightarrow A$ obeying the axioms on Fig.1. A *coalgebra* is object C equipped with counit $\epsilon = \epsilon_A : C \rightarrow \underline{1}$ and comultiplication $\Delta = \Delta_A : A \rightarrow A \otimes A$ obeying the axioms of algebra turned upside-down. Finally, [13],[14] a *bialgebra* A in a braided category \mathcal{C} is an object in \mathcal{C} equipped with algebra and coalgebra structures obeying the compatibility axiom on Fig.1 which means that Δ_A is an algebra homomorphism. A *Hopf algebra* A in a braided category \mathcal{C} (braided group or braided Hopf algebra) is a bialgebra in \mathcal{C} with antipode $S : A \rightarrow A$ which is convolution-inverse to identical map (the last identity on Fig.1). Axioms for (co-)module X over a (co-)algebra A are obtained by "polarization" of the (co-)algebra axioms.

If \mathcal{C} is a braided category we will denote by $\bar{\mathcal{C}}$ the same category with the same tensor product and with inverse braiding Ψ^{-1} . For any algebra (resp. coalgebra)

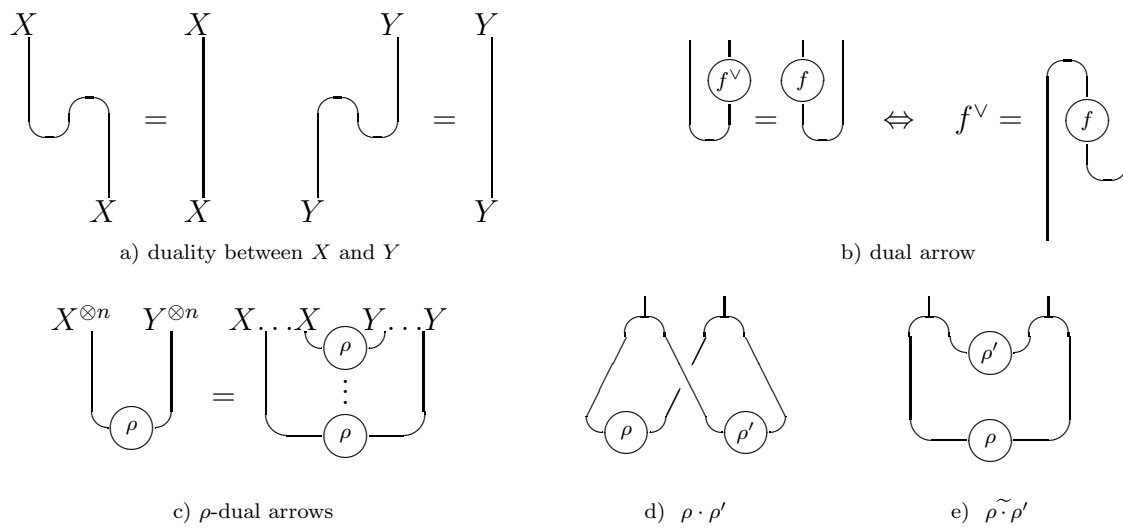


Figure 2: Duals and pairings.

A in \mathcal{C} we will always consider *the opposite algebra* $(A^{\text{op}}, \mu_{A^{\text{op}}} := \mu_A \circ \Psi^{-1})$ (resp. *the opposite coalgebra* $(A_{\text{op}}, \Delta_{A_{\text{op}}} := \Psi^{-1} \circ \Delta_A)$) as an object of the category $\overline{\mathcal{C}}$. In particular, $(A^{\text{op}})^{\text{op}} = A$. If A is a bialgebra in \mathcal{C} then A^{op} and A_{op} are bialgebras in $\overline{\mathcal{C}}$ (cf. [17]). Antipode S^- for A^{op} (or, the same, for A_{op}) is called *skew antipode* and equals S^{-1} if both S and S^- exist. Majid [17] derived from Hopf algebra axioms that antipode S_A is a bialgebra morphism $(A^{\text{op}})_{\text{op}} \rightarrow A$ (or $A \rightarrow (A_{\text{op}})^{\text{op}}$) in \mathcal{C} .

1.4 For objects X, Y of a monoidal category \mathcal{C} we will call any morphism

$$\cup = \cup^{X,Y} : X \otimes Y \rightarrow \mathbf{1} \quad (\text{ resp. } \cap = \cap_{Y,X} : \mathbf{1} \rightarrow Y \otimes X) \quad (3)$$

a *pairing between X, Y* (resp. *copairing between Y, X*). *Duality between X and Y* is both pairing and copairing (3) obeying the identities on Fig.2a. In this case X is called *left dual* to Y (resp. Y is called *right dual* to X) and we will write $X = {}^{\vee}Y, Y = X^{\vee}$. Dual arrow f^{\vee} is defined by one of the two equivalent conditions on Fig.2b. In this way a braided monoidal functor $(_)^{\vee} : \mathcal{C} \rightarrow \mathcal{C}_{\text{op}}^{\text{op}}$ can be defined if X^{\vee} exists for each $X \in \text{Obj}(\mathcal{C})$. Without loss of generality by coherence theorem we shall assume that $(_)^{\vee}$ is a strict monoidal functor: $(X \otimes Y)^{\vee} = Y^{\vee} \otimes X^{\vee}$, $(f \otimes g)^{\vee} = g^{\vee} \otimes f^{\vee}$. Pairing ρ between X and Y extends to pairing between $X^{\otimes n}$ and $Y^{\otimes n}$ defined by the diagram on Fig.2c. We say that arrows $f : X^{\otimes m} \rightarrow X^{\otimes n}$ and $g : Y^{\otimes n} \rightarrow Y^{\otimes m}$ are ρ -dual if $\rho \circ (f \otimes Y^{\otimes n}) = \rho \circ (X^{\otimes n} \otimes g)$.

Let A and H be bialgebras in braided category \mathcal{C} . Morphism $\rho : A \otimes H \rightarrow \mathbf{1}$ is called a *bialgebra pairing* if algebra (resp. coalgebra) structure on A and coalgebra (resp. algebra) structure on H are ρ -dual. Convolution product \cdot and 'the second' product \curvearrowright for $\rho, \rho' \in \text{Hom}_{\mathcal{C}}(X \otimes Y, \mathbf{1})$ are defined on Fig.2d,e. We denote by ρ^-, ρ^{\sim}

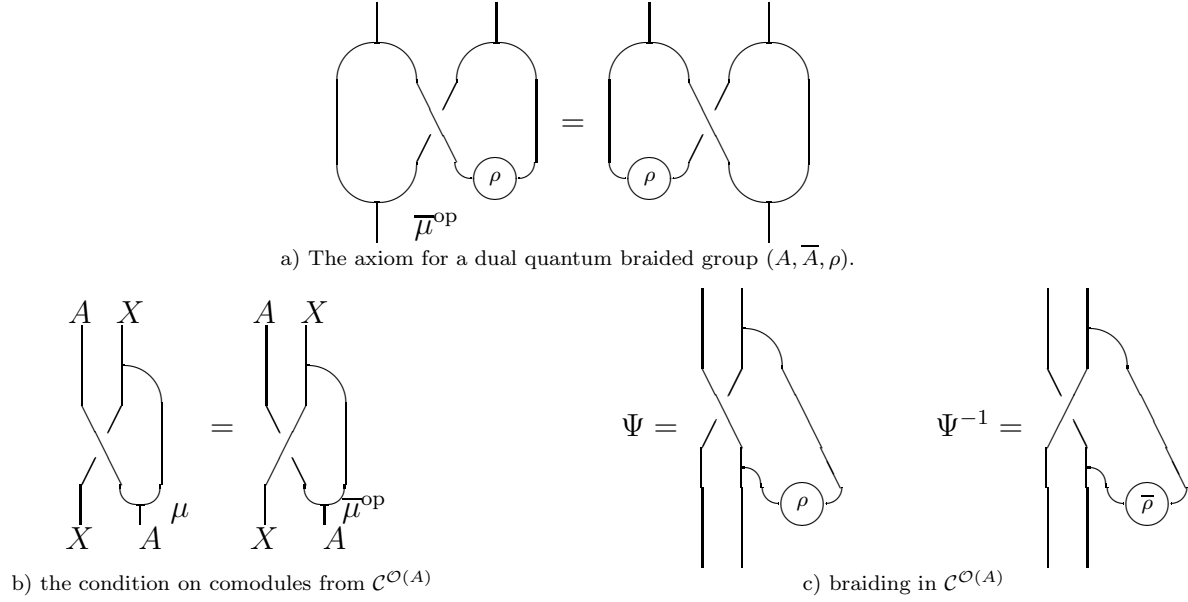


Figure 3:

corresponding inverse to ρ . Let $\bar{\rho} := \rho^- \circ \Psi^{-1}$. If A or H has (skew) antipode then ρ^\sim (resp. ρ^-) exists and

$$\rho \circ (S_A \otimes H) = \rho^\sim = \rho \circ (A \otimes S_H) \quad \rho \circ (S_A^- \otimes H) = \rho^- = \rho \circ (A \otimes S_H^-) \quad (4)$$

If ρ^- or ρ^\sim exists then ρ -duality between multiplications and comultiplications implies ρ -duality between units and counits. If (A, H, ρ) is bialgebra pairing in \mathcal{C} then $(A_{\text{op}}, H_{\text{op}}, \rho^-)$, $(A^{\text{op}}, H^{\text{op}}, \rho^\sim)$, $(H^{\text{op}}, A^{\text{op}}, \bar{\rho})$ are bialgebra pairings in $\overline{\mathcal{C}}$.

1.5 Quantum braided groups in a braided category were introduced in [14] and basic theory was developed there. The following are input-output reversed variant of definitions from [14] in a slightly modified form [3] suitable for our use.

A *coquasitriangular bialgebra* in a braided category \mathcal{C} is a pair of bialgebras A in \mathcal{C} and \overline{A} in $\overline{\mathcal{C}}$ with the same underlying coalgebra (μ and $\bar{\mu}$ are multiplications in A and \overline{A} respectively), and convolution invertible bialgebra pairing (*coquasitriangular structure*) $\rho : \overline{A}^{\text{op}} \otimes A \rightarrow \mathbb{1}$, satisfying the condition on Fig.3a. (It follows directly from the definition that units for A and for \overline{A} are the same.) A *dual quantum braided group* or a *coquasitriangular Hopf algebra* in \mathcal{C} is a coquasitriangular bialgebra such that A and \overline{A} have antipodes S and \overline{S} respectively. (In this case $\rho^- = \rho \circ (\overline{S} \otimes A)$ and $\rho^\sim = \rho \circ (A \otimes S)$.)

In particular, for any bialgebra (braided group) A the pair (A, A^{op}) is a coquasitriangular bialgebra (dual quantum braided group) with the trivial coquasitriangular structure $\rho = \epsilon \otimes \epsilon$.

Category $\mathcal{C}^{\mathcal{O}(A, \overline{A})}$ is a full subcategory of the category \mathcal{C}^A of right comodules with objects X satisfying the first identity on Fig.3b. $\mathcal{C}^{\mathcal{O}(A, \overline{A})}$ is a monoidal subcat-

egory of \mathcal{C}^A and braided with Ψ and Ψ^{-1} shown on Fig.3c. We use a brief notation $\mathcal{C}^{\mathcal{O}(A)}$ for $\mathcal{C}^{\mathcal{O}(A, A^{\text{op}})}$.

2 On braided FRT-construction

2.1 Canonical epimorphism $B_n \rightarrow S_n$ of the braid group into the permutation group admits a section $\widehat{S}_n \rightarrow B_n$ identical on generators and uniquely defined by condition that $\widehat{\sigma_1 \sigma_2} = \widehat{\sigma_1} \widehat{\sigma_2}$ if $\ell(\sigma_1 \sigma_2) = \ell(\sigma_1) + \ell(\sigma_2)$ where $\ell(\sigma)$ is the length (of the minimal decomposition) of σ . For any object X obvious action of braid group B_n on $X^{\otimes n}$ is defined. We will use the same notation $\widehat{\sigma}$ for the image of the braid $\widehat{\sigma} \in B_n$ in $\text{End}_{\mathcal{C}}(X^{\otimes n})$. For $k = 1, \dots, n$ let us denote by $S_n^k \subset S_n$ subset of $\frac{n!}{k!(n-k)!}$ shuffle permutations which preserves order of any two elements i and j if $i, j \leq k$ or $i, j > k$. Majid in [16] defines braided binomial coefficient as a sum of $\frac{n!}{k!(n-k)!}$ braids in $\text{End}_{\mathcal{C}}(X^{\otimes n})$ and in particular, braided factorial as a sum of $n!$ braids:

$$\left[\begin{matrix} n \\ k \end{matrix}; X \right] := \sum_{\sigma^{-1} \in S_n^k} \widehat{\sigma}, \quad [n; X]! := \sum_{\sigma \in S_n} \widehat{\sigma}. \quad (5)$$

2.2 For any object X of a braided category \mathcal{C} the tensor algebra $T(X) = \{X^{\otimes n}\}_{n \in \mathbb{N}}$ is a graded Hopf algebra with the tensor product as multiplication, comultiplication

$$\Delta_{m,n} := \left[\begin{matrix} m+n \\ m \end{matrix}; X \right] : X^{\otimes(m+n)} \rightarrow X^{\otimes m} \otimes X^{\otimes n} \quad (6)$$

and antipode

$$S|_{X^{\otimes n}} := (-1)^n \circ \widehat{\rho_n} : X^{\otimes n} \rightarrow X^{\otimes n}. \quad (7)$$

Where $S_n \ni \rho_n : (1, 2, \dots, n) \mapsto (n, n-1, \dots, 1)$ and $\widehat{\rho_n}$ is a Garside element of B_n . The bialgebra axiom turns into Newton-Majid binomial formula [16]:

$$(\underline{1} \otimes x + x \otimes \underline{1})^n = (\Delta \circ x)^n = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix}; X \right] \circ (x^{\otimes k} \otimes x^{\otimes(n-k)}) \quad (8)$$

for any $x : Z \rightarrow X$.

One can define two graded ideals in $T(X)$: $I = \{I_n \subset X^{\otimes n}\}$ is an ideal generated by its 'quadratic part' $I_2 := \ker [2; X]! = \ker(\Psi_{X,X} + \text{id})$. And let $I^\bullet = \{I_n^\bullet := \ker [n; X]!\}$.

I is non zero iff -1 is an eigenvalue of $\Psi_{X,X}$. If we suppose that the multiplicity of this eigenvalue is 1, i. e. we can choose a minimal polynomial of $\Psi_{X,X}$ in the form $p(t) = p_{-1}(t)(t+1)$ with $p_{-1}(-1) = 1$, then $P_{-1} := p_{-1}(\Psi_{X,X})$ is an idempotent, and in this case $I_2 = \ker [n; X]! = \text{im } P_{-1}$.

It's easy to see that $I \subset I^\bullet$. The following example from [5] show that, in general, the ideal I^\bullet has 'generators' of power more than 2.

Example 2.2.1 Let $T(X)$ be an algebra generated by the one dimensional vector space $X = kx$ over a field k with the braiding $\Psi(x \otimes x) = q(x \otimes x)$, $q \in k$. In this case braided integers are 'ordinary' q -integers: $[n]_q := 1 + q + \dots + q^{n-1}$. And for q a primitive root of 1 of order $n > 2$: $I = \emptyset$ but $I^\bullet = (x^{\otimes n})$.

Propositon 2.2.1 Both I and I^\bullet are Hopf ideals in $T(X)$. We denote by $V(X)$ and $V^\bullet(X)$ corresponding factor-algebras

For the special case of the category \mathcal{C} built from an arbitrary R -matrix $V(X)$ is an algebra of functions on 'quantum vector space'. Majid discovered a Hopf algebra structure on this object (cf. [16] and references therein).

2.3 It is well known that any solution $R = R_{X,X} : X \otimes X \rightarrow X \otimes X$, $X \in \text{Obj}(X)$ of the braid equation

$$(X \otimes R) \circ (R \otimes X) \circ (X \otimes R) = (R \otimes X) \circ (X \otimes R) \circ (R \otimes X) \quad (9)$$

with a certain invertibility conditions defines a braided structure on the monoidal subcategory of \mathcal{C} generated by object X and its dual ${}^{\vee}X$ as described in what follows. Morphisms $R_{X^{\otimes m}, X^{\otimes n}}$ are uniquely defined by the hexagon identities:

$$R_{Y \otimes Y', Z} = (R_{Y, Z} \otimes Y') \circ (Y \otimes R_{Y', Z}), \quad R_{Y, Z \otimes Z'} = (Z \otimes R_{Y, Z'}) \circ (R_{Y, Z} \otimes Z'). \quad (10)$$

where Y, Y', Z, Z' are powers of X . And let $R_{X^{\otimes m}, {}^{\vee}X^{\otimes n}} := {}^{\vee}(R_{X^{\otimes m}, X^{\otimes n}})$. We also suppose that there exists $R_{X, {}^{\vee}X} : X \otimes {}^{\vee}X \rightarrow {}^{\vee}X \otimes X$ inverse to $(\cup \otimes X \otimes {}^{\vee}X) \circ ({}^{\vee}X \otimes R_{X, X} \otimes {}^{\vee}X) \circ ({}^{\vee}X \otimes X \otimes \cap)$. Let $R_{X^{\otimes m}, {}^{\vee}X^{\otimes n}}$ be uniquely defined by the hexagon identities (10) and $R_{{}^{\vee}X^{\otimes m}, X^{\otimes n}}$ be defined in dual way. $\mathcal{C}(X, {}^{\vee}X; R)$ is a subcategory of \mathcal{C} whose objects are tensor products of X and ${}^{\vee}X$ and morphism $f : Y \rightarrow Z$ are those from \mathcal{C} which 'flow' along the braids labeled by $R_{X, -}$ and $R_{-, X}$, i.e.

$$R_{X, Z} \circ (X \otimes f) = (f \otimes X) \circ R_{X, Y}, \quad R_{Z, X} \circ (f \otimes X) = (X \otimes f) \circ R_{Y, X}. \quad (11)$$

The analog $A(X, R)$ of FRT-bialgebra [8] can be obtained as a result of a some type of reconstruction for the monoidal functor $\mathcal{C}(X, {}^{\vee}X; R) \rightarrow \mathcal{C}$.

2.4 As the first step the following lemmas allow us to define a bialgebra $A(X)$.

Lemma 2.4.1 Let X be an object of \mathcal{C} with (left) dual ${}^{\vee}X$. Then ${}^{\vee}X \otimes X$ can be equipped with a coalgebra structure

$$\Delta_{{}^{\vee}X \otimes X} := {}^{\vee}X \otimes \cap_{X, {}^{\vee}X} \otimes X, \quad \epsilon_{{}^{\vee}X \otimes X} := \cup^{{}^{\vee}X, X}. \quad (12)$$

X (resp. ${}^{\vee}X$) becomes a right (resp. left) comodule over ${}^{\vee}X \otimes X$ with coaction

$$\Delta_r^X := \cap_{X, {}^{\vee}X} \otimes X, \quad \Delta_\ell^{{}^{\vee}X} := {}^{\vee}X \otimes \cap_{X, {}^{\vee}X}. \quad (13)$$

Lemma 2.4.2 *Let $(A, \Delta_A), (B, \Delta_B)$ be coalgebras in \mathcal{C} and $(X, \Delta_r^X), (Y, \Delta_r^Y)$ right comodules over A and B respectively. Then $A \otimes B$ is a coalgebra with comultiplication*

$$\Delta_{A \otimes B} := (A \otimes \Psi_{A,B} \otimes B) \circ (\Delta_A \otimes \Delta_B) \quad (14)$$

and $X \otimes Y$ is a right $A \otimes B$ -comodule with coaction

$$\Delta_r^{X \otimes Y} := (X \otimes \Psi_{A,Y} \otimes B) \circ (\Delta_r^X \otimes \Delta_r^Y) \quad (15)$$

So with any two objects X and Y which have left duals we can connect the following coalgebras: tensor product of two coalgebras $({}^\vee X \otimes X) \otimes ({}^\vee Y \otimes Y)$ and coalgebra $({}^\vee Y \otimes {}^\vee X) \otimes (X \otimes Y)$ related with the object $X \otimes Y$.

Lemma 2.4.3 *Morphism*

$$\mu^{\vee X \otimes X, \vee Y \otimes Y} := \Psi_{\vee X \otimes X, \vee Y} \otimes Y : ({}^\vee X \otimes X) \otimes ({}^\vee Y \otimes Y) \rightarrow ({}^\vee Y \otimes {}^\vee X) \otimes (X \otimes Y) \quad (16)$$

is coalgebra isomorphism and interlaces coactions of these coalgebras on $X \otimes Y$.

For objects X, Y, Z with left dual the following associativity condition is true:

$$\mu_{X \otimes Y, Z} \circ (\mu_{X,Y} \otimes Z) = \mu_{X,Y \otimes Z} \circ (X \otimes \mu_{Y,Z}) \quad (17)$$

Propositon 2.4.4 $A(X) := \{A_n(X) = {}^\vee X^{\otimes n} \otimes X^{\otimes n}\}_{n \in \mathbb{Z}_{\geq 0}}$ *is a graded bialgebra with the following (co)multiplications:*

$$\begin{aligned} \mu_{m,n} &:= \mu^{\vee X^{\otimes m} \otimes X^{\otimes m}, \vee X^{\otimes n} \otimes X^{\otimes n}}, \\ \Delta_n &:= \Delta_{\vee X^{\otimes n} \otimes X^{\otimes n}} : {}^\vee X^{\otimes n} \otimes X^{\otimes n} \rightarrow ({}^\vee X^{\otimes n} \otimes X^{\otimes n}) \otimes ({}^\vee X^{\otimes n} \otimes X^{\otimes n}). \end{aligned}$$

Graded tensor algebra $T(X)$ (resp. $T({}^\vee X)$) is right (resp. left) $A(X)$ -comodule algebra and coalgebra.

One can carry out the same construction in the category $\overline{\mathcal{C}}$. The result is a bialgebra $\overline{A}(X)$ with the same underlying coalgebra but with new multiplication $\overline{\mu}$ where Ψ is replaced by Ψ^{-1} . Corresponding Hopf algebra $\overline{T}(X)$ (resp. $\overline{V}(X), \overline{V}^\bullet(X)$) coincide with $T(X)_{\text{op}}$ (resp. $V(X)_{\text{op}}, V^\bullet(X)_{\text{op}}$).

2.5 Let, moreover, duality C of X with itself be given. Then for each n the pairing and copairing

$$C = C^{X^{\otimes n}, X^{\otimes n}} : X^{\otimes n} \otimes X^{\otimes n} \rightarrow \overline{1}, \quad C = C_{X^{\otimes n}, X^{\otimes n}} : \overline{1} \rightarrow X^{\otimes n} \otimes X^{\otimes n}, \quad (18)$$

described by the diagram on Fig.2b and by the input-output reversed diagram, define duality of $X^{\otimes n}$ with itself. Let as define pairing ${}^\vee C =: {}^\vee X^{\otimes n} \otimes {}^\vee X^{\otimes n} \rightarrow \overline{1}$ as morphism left dual to copairing $C_{X^{\otimes n}, X^{\otimes n}}$. We denote by $A(X; C)$ filtered algebra which is a factor-algebra of $A(X)$ by the ideal 'generated by relations' on Fig.4 which means that pairings C and ${}^\vee C$ are invariant with respect to coactions of $A(X)$ on $T(X)$ and $T({}^\vee X)$ respectively. And let $T(X; C)$ be factor-algebra of $T(X)$ by relations $C^{\vee X \otimes X} = \underline{1}$.

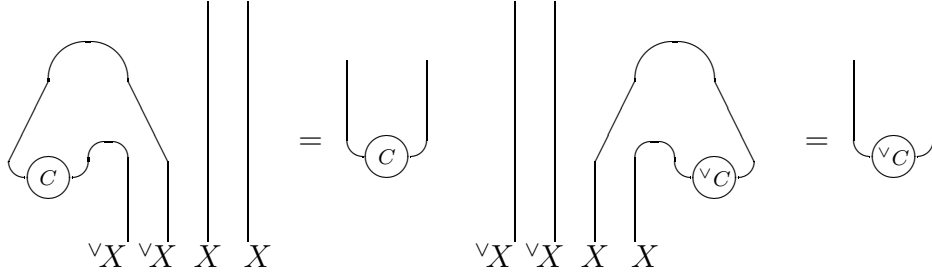
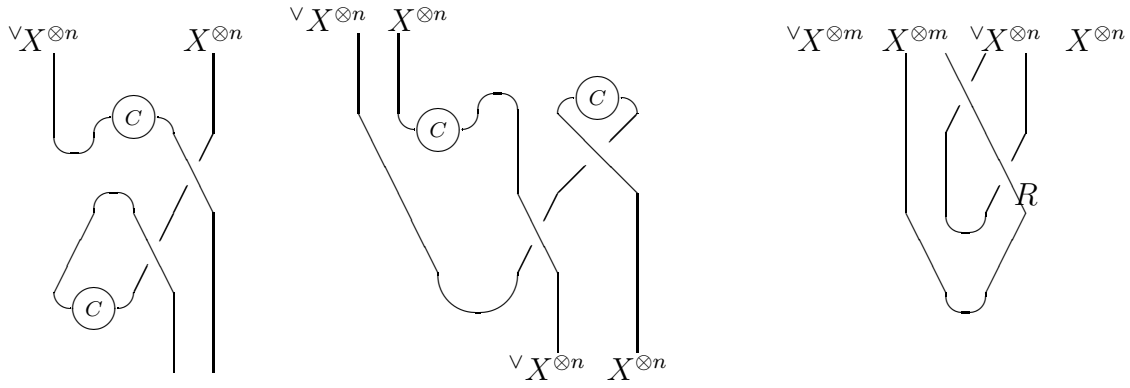


Figure 4: Relations for $A(X; C)$



a) antipode and its inverse for $A(X; C)$

b) Coquasitriangular structure on $A(X, R)$

Figure 5:

Propositon 2.5.1 $A(X; C)$ is a braided group with antipode and its inverse given by the diagram on Fig5a. $T(X; C)$ is a right comodule algebra over $A(X; C)$.

Let $I = \{I_n \in {}^\vee X^{\otimes n} \otimes X^{\otimes n}\}$ be a graded ideal of algebra $A(X)$ generated by relations

$${}^\vee R \otimes \text{id}_{X^{\otimes 2}} - \text{id}_{{}^\vee X^{\otimes 2}} \otimes R : {}^\vee X^{\otimes 2} \otimes X^{\otimes 2} \rightarrow {}^\vee X^{\otimes 2} \otimes X^{\otimes 2} X^{\otimes 2} \otimes X^{\otimes 2} \quad (19)$$

or explicitly

$$I_n = \bigcup_{i=1}^{n-1} ({}^\vee X^{\otimes(i-1)} \otimes {}^\vee R \otimes {}^\vee X^{\otimes(n-i-1)} \otimes X^{\otimes n} \rightarrow {}^\vee X^{\otimes n} \otimes X^{\otimes(n-i-1)} \otimes R \otimes X^{\otimes(i-1)}) \quad (20)$$

Lemma 2.5.2 Ideal I described above is a biideal of $A(X)$. We denote by $A(X, R)$ corresponding factor-bialgebra.

Let, moreover, C be morphism in $\mathcal{C}(X, {}^\vee X; R)$, i.e. pairing C 'flows' along braids labeled by R . Then we define a bialgebra $A(X, R; C)$ which is a factor-algebra of $A(X)$ by an ideal generated by both (19) and the relations given on Fig.4. Similarly, one can define the factor-algebras $\overline{A}(X, R^{-1})$ and $\overline{A}(X, R^{-1}; C)$ the bialgebra $\overline{A}(X)$ in the category $\overline{\mathcal{C}}$.

Lemma 2.5.3 A family of pairings $\rho_{m,n} := ({}^\vee X^{\otimes m} \otimes X^{\otimes m}) \otimes ({}^\vee X^{\otimes m} \otimes X^{\otimes m}) \rightarrow \mathbb{1}$ described by the diagram on Fig.5b define bialgebra pairings

$$\begin{aligned} \rho_{A(X,R)} : \overline{A}(X, R^{-1})^{\text{op}} \otimes A(X, R) &\rightarrow \overline{\mathbb{1}}, \\ \rho :_{A(X,R;C)} \overline{A}(X, R^{-1}; C)^{\text{op}} \otimes A(X, R; C) &\rightarrow \overline{\mathbb{1}}. \end{aligned} \quad (21)$$

Theorem 2.5.4 $(A(X, R), \overline{A}(X, R^{-1}), \rho)$ is a coquasitriangular bialgebra and its factor-algebra $(A(X, R; C), \overline{A}(X, R^{-1}; C), \rho)$ is a dual quantum braided group in \mathcal{C} (or, more precisely, in a certain category of 'graded spaces' over \mathcal{C}). 'Second inverse' $\rho^\sim = \{\rho_{m,n}^\sim\}$ to quasitriangular structure ρ takes the form:

$$\rho_{m,n}^\sim := (\cup \otimes \cup) \circ ({}^\vee X^{\otimes m} \otimes \Psi_{X^{\otimes m}, {}^\vee X^{\otimes n}}^{-1} \otimes X^{\otimes n})$$

Lemma 2.5.5 $X^{\otimes n}$ equipped with coaction (13) is an object of $\mathcal{C}^{\mathcal{O}(A(X,R), \overline{A}(X,R^{-1}))}$. Braiding $\Psi_{X^{\otimes m}, X^{\otimes n}}$ in this category equals to $R_{X^{\otimes m}, X^{\otimes n}}$. Corresponding right action (defined by the first diagram on Fig.6b)

$$\mu_r^{X^{\otimes n}} = \{\mu_{r,m}^{X^{\otimes n}} : X^{\otimes n} \otimes A(X, R)_m \rightarrow X^{\otimes n}\}$$

takes the form

$$\mu_{r,m}^{X^{\otimes n}} = (\cup^{{}^\vee X^{\otimes m}, X^{\otimes m}} \otimes \text{id}_{X^{\otimes n}}) \circ (\text{id}_{{}^\vee X^{\otimes m}} \otimes R_{X^{\otimes n}, X^{\otimes m}}) \circ (\Psi_{X^{\otimes n}, {}^\vee X^{\otimes m}} \otimes \text{id}_{X^{\otimes m}}). \quad (22)$$

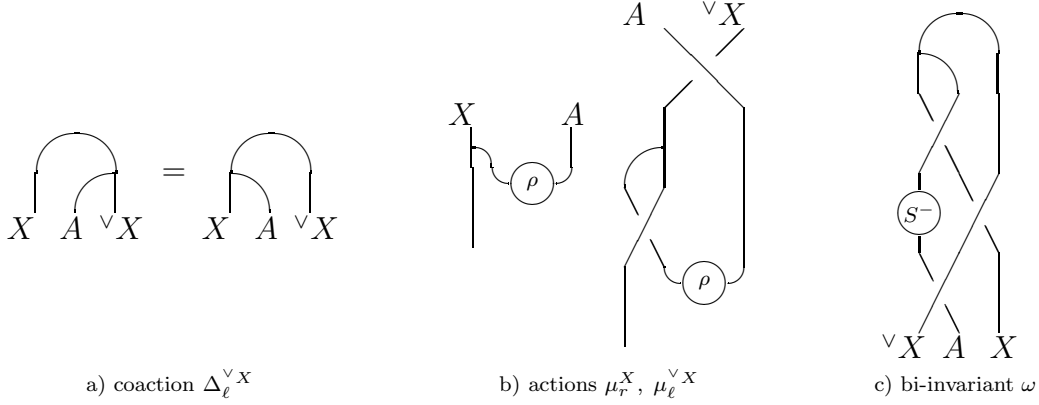


Figure 6:

One can carry out constructions from 2.2 for $X \in \text{Obj}(\mathcal{C}^{\mathcal{O}(A(X,R), \overline{A}(X,R^{-1}))})$ to get the Hopf algebras $T(X, R), V(X, R), V^\bullet(X, R)$ in $\mathcal{C}^{\mathcal{O}(A(X,R), \overline{A}(X,R^{-1}))}$, where letter 'R' is added to specify a category. A pair $(V(X, R), V(X, R)^{\text{op}})$ is a quantum braided group in $\mathcal{C}^{\mathcal{O}(A(X,R), \overline{A}(X,R^{-1}))}$ with the trivial coquasitriangular structure $\epsilon^{\nabla(X,R)} \otimes \epsilon_{V(X,R)}$. Generalized bosonization theorem [3] allows us to define a quantum braided group $(A(X, R) \ltimes V(X, R), \overline{A}(X, R^{-1}) \ltimes V(X, R)^{\text{op}})$ with coquasitriangular structure $(\text{id}_{A(X,R)} \otimes \epsilon_{V(X,R)} \otimes \text{id}_{A(X,R)} \otimes \epsilon_{V(X,R)}) \circ \rho_{A(X,R)}$ in \mathcal{C} which is an analog of algebra of functions on inhomogeneous linear group. The same construction performed for algebras $\overline{A}(X, R^{-1})$ and $\overline{V}(X, R^{-1}) = V(X, R)_{\text{op}}$ produces quantum braided group in $\overline{\mathcal{C}}$. But in this way we obtain another corresponding quantum braided group in \mathcal{C} .

2.6 Let (A, \overline{A}, ρ) be a dual quantum braided group in \mathcal{C} , $(X, \Delta_r^X) \in \text{Obj}(\mathcal{C}^{\mathcal{O}(A, \overline{A})})$, ∇X left dual to X in \mathcal{C} with left comodule structure $\Delta_\ell^{\nabla X}$ defined by the condition in Fig.6a. Then X (resp. ∇X) equipped with right (resp. left) A -module structure as shown in Fig.6b becomes right (resp. left) crossed module over A . According to general theory [4] the object $\Gamma := \nabla X \otimes A \otimes X$ with actions and coactions

$$\begin{aligned} \mu_\ell^\Gamma &:= (\mu_\ell^{\nabla X} \otimes m_A) \circ (A \otimes \Psi_{A, \nabla X} \otimes A) \circ (\Delta_A \otimes \nabla X \otimes A) \otimes X, \\ \mu_r^\Gamma &:= \nabla X \otimes (m_A \otimes \mu_r^X) \circ (A \otimes \Psi_{X, A} \otimes A) \circ (A \otimes X \otimes \Delta_A), \\ \Delta_\ell^\Gamma &:= (m_A \otimes \nabla X \otimes A) \circ (A \otimes \Psi_{\nabla X, A} \otimes A) \circ (\Delta_\ell^{\nabla X} \otimes \Delta_A) \otimes X, \\ \Delta_r^\Gamma &:= \nabla X \otimes (A \otimes X \otimes m_A) \circ (A \otimes \Psi_{A, X} \otimes A) \circ (\Delta_A \otimes \Delta_\ell^X). \end{aligned}$$

is a Hopf bimodule over A . Morphism ω defined on Fig.6c is a bicomodule morphism where $\underline{1}$ is equipped with trivial left and right actions equal to ϵ_A . 'Commutant with ω ':

$$d := \mu_r^\Gamma \circ (\omega \otimes A) - \mu_\ell^\Gamma \circ (A \otimes \omega) : A \rightarrow \Gamma \quad (23)$$

is a first order bicovariant derivative in sense of Woronowicz [20] (See [5] for a fully braided context).

In our case $A = A(X, R; C)$ 'biinvariant' ω equals to $\Psi_{\vee X, \vee X}^{-1} \circ \vee C \otimes C : \underline{1} \rightarrow \vee X^{\otimes 2} \otimes X^{\otimes 2}$.

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